Folklore problems

collected by Josef Tkadlec

Algebra

Problem A1. Prove that if $a + b + c = 8$, then

$$\sqrt{1 + a^2} + \sqrt{4 + b^2} + \sqrt{9 + c^2} \geq 10$$

for $a, b, c \in \mathbb{R}$.

Problem A2. Minimize

$$|x - 1| + |x - 2| + |x - 3| + |x - 100|$$

for $x \in \mathbb{R}$.

Problem A3. Let $a, b, c \in \mathbb{R}$, prove

$$a^2 + b^2 + c^2 \geq ab + bc + ca.$$

Problem A4. Let $n \in \mathbb{N}$, prove

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)} < 1.$$

Problem A5. Real numbers $x_1, \ldots, x_n$ satisfy for each $i = 1, \ldots, n$

$$x_i = \sum_{j \neq i} \frac{1}{x_i - x_j}.$$

Futhermore $x_1^2 + x_2^2 + \ldots + x_n^2 = 36$. Determine n.

Problem A6. Prove that the number $\left(5 + \sqrt{26}\right)^n$ has immediatelly after the decimal point at least n identical digits.

Problem A7. For every positive integer n, prove that

$$(1 + 2 + ... + n)^2 = 1^3 + 2^3 + ... + n^3.$$

Problem A8. Prove that the number one can be obtained as the sum of inverses of n mutually different positive integers for every $n \geq 3$.

Problem A9. Calculate

$$\sum_{n=1}^{99} \frac{1}{2\lfloor\sqrt{n}\rfloor + 1}.$$

Problem A10. Jack has taken four real numbers and calculated products of all the pairs. He obtained $2, 3, 4, 5, 6, x$ in some order. Determine x?

Problem A11. Let $m * n$ denote $\frac{m+n}{mn+4}$. Determine

$$((\ldots((2013 * 2012) * 2011) * \ldots * 2) * 1) * 0.$$

Problem A12. Prove that there are two irrational numbers a, b so that a^b is rational.

Problem A13. Let real number $x \neq 1$ satisfy

$$(x^2 + x + 1)(x^6 + x^3 + 1) = \frac{10}{x - 1}.$$

Determine x^{36}.

Problem A14. Positive real numbers x, y, z satisfy

$$\begin{aligned} x^2 + y^2 + xy &= 1, \\ y^2 + z^2 + yz &= 4, \\ z^2 + x^2 + zx &= 5. \end{aligned}$$

Determine $x + y + z$.

Geometry

Problem G1. Segments BC, CD of a square $ABCD$ contain points X, Y respectively so that $\angle XAY = 45°$. Segments AX, AY intersect the diagonal BD in points K, L respectively. Prove that points K, X, C, Y, L are concyclic.

Problem G2. Segments BC, CA, AB of a triangle contain points D, E, F respectively so that $|AF| = 2 \cdot |FB|$, $|BD| = 2 \cdot |DC|$, and $|CE| = 2 \cdot |EA|$. The lines AD, BE, CF define the boundaries of a triangle KLM. What is the ratio between areas of triangles KLM and ABC?

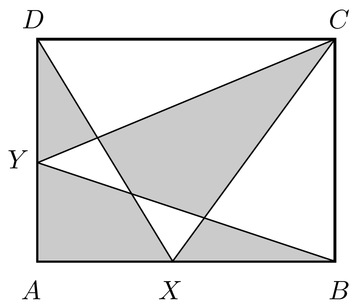
Problem G3. Show that it is possible to color every point in a three-dimensional space by one of four colors so that every color is used, and every line contains at most two colors.

Problem G4. Every point in a plane is colored by one of two colors. Prove that there is an equilateral triangle with monochromatic vertices.

Problem G5. Let D, E denote the touch points of the incircle with the segments BC, CA respectively in a triangle ABC. Prove that the angle bisector at the vertex A intersects the line DE on (some) midsegment of ABC.

Problem G6. Let AD be the altitude against the hypotenuse in a right triangle ABC. Let r_1, r_2, r denote the inradii of triangles ABD, ACD, ABC respectively. Prove $r_1^2 + r_2^2 = r^2$.

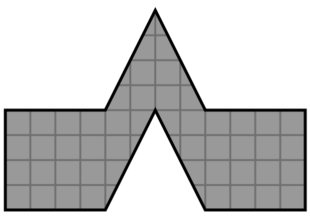
Problem G7. Points X, Y are given inside segments AB, AD of a restangle $ABCD$ respectively. Inside the rectangle, we draw segments XC, XD, YB, YC and fill the interior regions by white and grey colors in an alternating way so that the triangles adjacent to segments BC and CD are white. Prove that one of the grey areas are the sum of the remaining grey areas.



Problem G8. We have wooden stakes, metal rings, ropes, and one goat. The end of every rope have to be tied to the goat, stake on a fixed position, or to the metal ring. A rope can also pass through the metal ring. The goat will overgraze the grass at every point it can achieve. Wind such a setup of stakes, rings and ropes so that the overgrazed area will have the shape of a quarter of a circle.

Problem G9. A duck is swimming in a circular pond. There is a four times faster lurking fox on the circumference which is however afraid of entering the water. The duck can take off on the land only. Will the duck manage to escape the fox?

Problem G10. Divide the following shape into 7 congruent parts. (The grid is here for scale only)



Problem G11. Find a hexagon which can be divided by a single line into four congruent parts.

Problem G12. A quadrilateral $ABCD$ is inscribed into a circle of radius 5. Assume that $AB = 6$, $BC = 7$, $CD = 8$. Determine the length DA.

Combinatorics

Problem C1. Consider a square 8×8 with two opposite corners 1×1 removed. Prove that it cannot be tiled by rectangles 2×1 and 1×2.

Problem C2. Consider a square 32×32 with an individual 1×1 square removed (with integer coordinates). Prove that the remaining shape can be tiled by L-shaped trimina.

Problem C3. We have a rectangle chocolate bar $n \times m$. In each step, we can break one piece by the horizontal, or vertical line. Prove that the number of steps required to break the chocolate to individual pieces is independent of the breaking procedure.

Problem C4. Prove that every set of integers of size 100 has a non-empty subset sum of which is divisible by 100.

Problem C5. Let S be an $(n + 1)$-element subset of $\{1, 2, \ldots, 2n\}$. Prove that there are two distinct elements $a, b \in S$ such that a is divisible by b.

Problem C6. Prove that every coloring of edges of the complete graph on 6 vertices by two colors contains a monochromatic triangle.

Problem C7. Prove that every graph on at least two vertices contains two vertices of the same degree.

Problem C8. Prove that the edges of a complete graph on six vertices cannot be drawn by a single trail without repetition of edges.

Problem C9. In a South-African tribe Club-Club, every tribesman has a blue or red spot drawn on his forehead. According to an ancient tradition, whenever someone figures out the color of his dot, he has to commit suicide the next day by jumping out of the cliff. Once, a foreigner visits the village and on a public meeting reveal a fact: “At least one citizen has a blue dot.” Prove that he launches a series of suicides ending by the death of all the tribesmen.

Problem C10. There are 100 prisoners standing in a line, they have hats of two colors, every prisoner sees the hats of the prisoners in front of him. First the prisoner at the back is asked for a guess of the color of his hat, and then the examiner proceeds forward one by one. Every prisoner hears the previous guesses. If the prisoners can settle on a strategy beforehand, what is the maximal number of certainly correct answers?

Problem C11. There are 10 prisoners in a room, each of them has a hat, and every hat can have one of 10 colors (they can repeat). Every prisoner sees the colors of hats of the other prisoners but not of his own. At one moment, every prisoner makes a guess about the color of his hat. They can arrange a strategy beforehand. Find a strategy for them so that at least one prisoner will guess the right color, whatever the distribution of the colors is.

Problem C12. There are ten points randomly chosen on a circle. What is the probability that all of them are inside a single half circle?

Problem C13. There are several ants walking on a levitating 1 meter long string. When an ant reaches the end of the string, it falls down. The ants are walking with a constant speed 1 meter per minute in one direction, and if two ants meet each other, they “bounces” of each other, and continue with the opposite directions. Prove that after a minute, all the ants will have been out of the string.

Number theory

Problem N1. Find all the triples of integers satisfying the equation $x^3 + 2y^3 + 4z^3 = 8xyz$.

Problem N2. Find all pairs of positive integers (a, b) satisfying $a \mid 2b + 1$ and $b \mid 2a + 1$.

Problem N3. Let n be a positive integer. Prove that $n^2 + n + 1$ is not a square of an integer.

Problem N4. Find all pairs of integers (a, b) satisfying $ab = 2a + 3b$.

Problem N5. Show that there are 1000 consecutive composite numbers.

Problem N6. What is the sum of the (positive) divisors of the number 700?

Problem N7. Find all integer pairs (n, a) satisfying

$$1! + 2! + \cdots + n! = a^2.$$

Problem N8. Prove that the fraction

$$\frac{21n + 4}{14n + 3}$$

cannot be reduced for any positive integer n.

Problem N9. Prove that $\sqrt{2}$ is irrational.

Problem N10. Prove that there are infinitely many prime numbers.

Problem N11. Let n be a positive integer. Is $n^2 - n + 41$ necessarily a prime number?

Problem N12. Determine the largest integer n such that $(n!!!)$ divides $(2019!!)$.

Problem N13. The sum of several (at least two) consecutive integers is 100. What is the biggest number that could be among them?

Hints

Hint A1. Triangle inequality

Hint A3. $(a-b)^2+(b-c)^2+(c-a)^2 \geq 0$

Hint A4. $1-\frac{1}{n+1}$

Hint A5. Multiply i-th equation by x_i and sum up.

Hint A6. $\left(5+\sqrt{26}\right)^n+\left(5-\sqrt{26}\right)^n$ is an integer

Hint A7. By induction.

Hint A8. Induction step: Divide by two and add 1/2.

Hint A11. $x*2=1/2$

Hint A12. $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}=2$. Use excluded middle.

Hint A14. Cosine theorem, right triangle $1,2,\sqrt{5}$, equilateral triangle adjacent to a leg, $x+y+z$ is the distance of the two vertices that are not shared by the two triangles.

Hint G3. $d+1$ colors for d dimensions

Hint G6. Pythagorean theorem and simillarity

Hint G10. stripes

Hint C2. Compose 2 times larges trimino from 4 trimina, and iterate.

Hint C3. Every break increases the number of pieces by one.

Hint N4. $(a-3)(b-2)=6$

Hint N5. $1001!+2,\ldots,1001!+1001$

Hint N7. Modulo 5 for ruling out large n.

Answers

Answer A2. 100

Answer A5. 9

Answer A9. 9

Answer A10. 12/5

Answer A13. 14641

Answer A14. $\sqrt{5+2\sqrt{3}}$

Answer G2. 1/7

Answer C10. 99

Answer C12. $10/2^{10}$

Answer N1. $(0,0,0)$

Answer N2. $(1,1),(1,3),(3,1),(3,7),(7,3)$

Answer N4. $(4,8),(5,5),(6,4),(9,3),(2,-4),(1,-1),(0,0),(-3,1)$

Answer N6. 1736

Answer N7. $(1,1),(3,3)$

Answer N11. No.

Answer N12. 6

Answer N13. 202